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On Mean Field Equations for Spin Glasses

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In this paper we study rigorously the random Ising model on a Cayley tree in the limit of infinite coordination number $z \to \infty$. An iterative scheme is developed relating mean magnetizations and mean square magnetizations of successive shells far removed from the surface of the lattice. In this way we obtain local properties of the model in the (thermodynamic) limit of an infinite number of shells. When the coupling constants are independent Gaussian random variables the SK expressions emerge as stable fixed points of our scheme and provide a valid local mean-field theory of spin glasses in which negative local entropy (at low temperatures) while perfectly possible mathematically may still perhaps be physically undesirable. Finally we examine the TAP equations and show that if the average over bond disorder and the limit $z \to \infty$ are actually performed, one recovers our iterative scheme and hence the SK equations also in the thermodynamic limit.

KEY WORDS: Cayley tree; iteration; fixed point; spin glass; Gaussian distribution; local mean-field theory; SK equations; TAP equations.

1. INTRODUCTION

Since the original work of Edwards and Anderson[1] and Sherrington and Kirkpatrick (SK),[2] much has been written about the validity of these authors' mean-field theories for spin glasses (Refs. 3–5 and references quoted therein).

The random Curie–Weiss model considered by SK and the resulting mean-field-type equations have a certain appeal to them but, as pointed out by SK, these equations lead to negative entropy at low temperatures. Use of the so-called n-replica trick and interchanges of limits have been suggested as causes of this unphysical phenomenon and various theories have

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been proposed to overcome these difficulties. It is probably fair to say, however, that there is still no universally acceptable mean-field theory of spin glasses at this time, nor are there any rigorous results on the range of validity of the SK equations.

An alternative approach to the problem was proposed by Thouless, Anderson, and Palmer (TAP)[6] in which the n-replica trick was avoided by taking thermal averages at the outset on a lattice with large coordination number z. The resulting TAP equations apparently lead to a physically acceptable entropy but there are still difficulties in performing the average over bond disorder and controlling the mean-field limit $z \to \infty$.

In their derivation, TAP argue that for large z the only terms in a graphical expansion that are important are those corresponding to a Bethe lattice or Cayley tree. On the other hand, Katsura, Inawashiro, and Fujiki[7] consider a variant of the Bethe approximation, and also without use of the replica trick, obtain the SK equations in the limit $z \to \infty$.

In order to understand this apparent discrepancy we present here a rigorous analysis of the random Ising model on a Cayley tree in the limit of infinite coordination number z. What we obtain in this limit is an iterative scheme

$$\begin{aligned} m_{i+1} &= f(m_i, q_i) \\ q_{i+1} &= g(m_i, q_i) \end{aligned} \tag{1.1}$$

relating the mean magnetizations m_i and their mean squares q_i in layers or shells i of the tree far removed from the surface. In the limit $i \to \infty$, m_i and q_i converge to a stable fixed point m, q of (1.1) which then become equivalent to the SK equations.

Since surface effects have been eliminated in the limit $i \to \infty$, it is clear that m and q should be interpreted as *local* quantities rather than global or bulk expressions for the magnetization and its mean square, respectively. This is also the case for the nonrandom Ising model on a Cayley tree,[8] where it is known that the bulk properties[9] are not described by local Bethe approximation expressions.

The free energy obtained from the SK equations should likewise be considered a *local* free energy, at least for the Cayley tree model. Negative local entropy is then mathematically possible but probably still physically undesirable. Nevertheless the random Ising model on a Cayley tree provides an instance where the SK equations are rigorously valid and also provides a possible interpretation of this theory as a local mean-field theory.

Our analysis certainly does not provide an exact solution to the original SK model nor does it provide a global mean-field theory of the spin glass state.

In the following section we formulate the general Ising problem on a Cayley tree and derive expressions for local quantities. In Section 3 we perform averages with respect to a Gaussian distribution of bond strengths in the infinite coordination number ($z \to \infty$) limit. In this limit we obtain an "SK hierarchy" (1.1) from which the SK expressions emerge as stable fixed points. In Section 4 we reexamine the TAP equations and show that if averaging over bond disorder and the limit $z \to \infty$ are actually performed one also recovers the hierarchy (1.1) from the TAP equations. Our results are summarized and discussed in the final section.

2. GENERAL FORMULATION OF THE ISING MODEL ON A CAYLEY TREE

Since the model has been formulated in detail elsewhere,[8,10] we will simply summarize the pertinent points and equations here.

We consider a Cayley tree with N shells and coordination number z. (The case $N = 2$ and $z = 3$ is shown in Fig. 1.) Starting from a central spin $\sigma_0 = \pm 1$ we label spins in successive shells $s = 1, 2, \ldots, N$ with the indices $(s) \equiv (i_1 i_2 \ldots i_s)$ $i_1 = 1, 2, \ldots, z$; $i_k = 1, 2, \ldots, z-1$ for $k = 2, \ldots, s$. Assuming nearest neighbor interactions only and a uniform magnetic field H, the interaction energy $E\{\sigma\}$ in a given configuration of spins $\{\sigma_{(s)} = \pm 1\}$ is given by

$$
\begin{aligned}
E\{\sigma\} = &- \sum_{i_1=1}^{z} J_{0,i_1}\sigma_0\sigma_{i_1} - \sum_{i_1=1}^{z}\sum_{i_2=1}^{z-1} J_{i_1,i_1i_2}\sigma_{i_1}\sigma_{i_1i_2} - \cdots \\
&- \sum_{i_1=1}^{z}\sum_{i_2=1}^{z-1}\cdots\sum_{i_N=1}^{z-1} J_{i_1\ldots i_{N-1},i_1\ldots i_N}\sigma_{i_1\ldots i_{N-1}}\sigma_{i_1\ldots i_N} \\
&- H\Bigg(\sigma_0 + \sum_{i_1=1}^{z}\sigma_{i_1} + \sum_{i_1=1}^{z}\sum_{i_2=1}^{z-1}\sigma_{i_1i_2} \\
&+ \cdots + \sum_{i_1=1}^{z}\sum_{i_2=1}^{z-1}\cdots\sum_{i_N=1}^{z-1}\sigma_{i_1i_2\ldots i_N}\Bigg)
\end{aligned} \tag{2.1}
$$

By summing successively over spins in shells $s = 1, 2, \ldots, N$, we obtain the following expression for the partition function:

$$
Z_N = \sum_{\{\sigma\}} \exp(-\beta E\{\sigma\}) = \prod_{s=1}^{N} T_s \tag{2.2}
$$

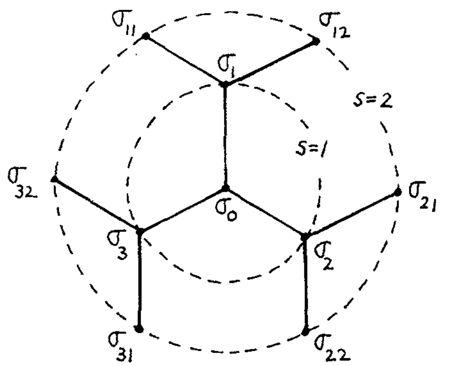

Fig. 1. Two shells of a Cayley tree with coordination number 3.

where

$$T_s = \begin{cases} e^B \prod_{i_1=1}^{z} 2\cosh(K_{0,i_1} + L_{i_1}) + e^{-B} \prod_{i_1=1}^{z} 2\cosh(K_{0,i_1} - L_{i_1}), & s = 1 \\ \prod_{i_1=1}^{z} \prod_{i_2=1}^{z-1} \cdots \prod_{i_s=1}^{z-1} \big[4\cosh(K_{(s-1),(s)} + L_{(s)}) & \\ \qquad \times \cosh(K_{(s-1),(s)} - L_{(s)})\big]^{1/2}, & s \neq 1 \end{cases} \tag{2.3}$$

$$K_{(s-1),(s)} = \beta J_{i_1 \dots i_{s-1}, i_1 \dots i_s}, \qquad B = \beta H \tag{2.4}$$

and the $L_{(s)} = L_{i_1 i_2 \dots i_s}$ are defined recursively by

$$L_{(s)} = B + \sum_{i_{s+1}=1}^{z-1} \operatorname{artanh}(\tanh K_{(s),(s+1)} \tanh L_{(s+1)}) \tag{2.5}$$

with "boundary condition"

$$L_{(N)} = B \tag{2.6}$$

The thermal expectation value for the central spin σ_0 is given by

$$\begin{aligned} \langle \sigma_0 \rangle &\equiv Z_N^{-1} \sum_{\{\sigma\}} \sigma_0 \exp(-\beta E\{\sigma\}) \\ &= \tanh\left[B + \sum_{i=1}^{z} \operatorname{artanh}(\tanh K_{0,i} \tanh L_i) \right] \end{aligned} \tag{2.7}$$

and for later reference, the thermal expectation value of a spin σ_j in the first shell is given by

$$\begin{aligned} \langle \sigma_j \rangle &= \tfrac{1}{2}\big[\tanh(K_{0,j} + L_j) - \tanh(K_{0,j} - L_j)\big] \\ &\quad + \tfrac{1}{2}\big[\tanh(K_{0,j} + L_j) + \tanh(K_{0,j} - L_j)\big]\langle \sigma_0 \rangle \qquad (j = 1, 2, \dots, z) \end{aligned} \tag{2.8}$$

One major difficulty or drawback of the Cayley tree model is the unusual influence of boundary or surface effects. This is already apparent in the above equations if one sets the field H equal to zero at the outset. Thus from (2.6) and (2.7), the $L_{(s)}$ are all zero and (2.2) collapses into what amounts to a one-dimensional Ising model.[11] Also a simple calculation shows that the number of spins in shell s is given by

$$\nu_s = z(z-1)^{s-1} \tag{2.9}$$

so that of the total number of spins for N shells,

$$\nu(N) = 1 + \sum_{s=1}^{N} \nu_s = (z-2)^{-1}\left[z(z-1)^N - 2\right] \tag{2.10}$$

a finite fraction

$$\nu_N/\nu(N) \sim (z-2)(z-1)^{-1} \qquad \text{as } N\to\infty \tag{2.11}$$

remain on the boundary.

It is precisely because of this feature of the Cayley tree that the bulk properties of the model display a peculiar type of phase transition[9] that bears no relation to the Bethe approximation.

What is true, however, is that *local* properties, such as the expectation values (2.7) and (2.8) of spins "deep inside" the lattice (in the limit $N\to\infty$) are in agreement with the Bethe approximation.[8]

Here we are interested in local properties of the Ising model on a Cayley tree when the coupling constants $J_{(s),(s+1)}$ are random variables. In particular, we evaluate (quenched) averages of expressions (2.7) and (2.8) in the following sections for the case of independent Gaussian random variables $J_{(s),(s+1)}$ in the limit $z\to\infty$.

3. GAUSSIAN DISTRIBUTION OF COUPLING CONSTANTS IN THE LIMIT $z\to\infty$

From (2.5), (2.6), and (2.7) we have the hierarchy

$$\begin{aligned}
\langle\sigma_0\rangle &= \tanh\left[B + \sum_{i=1}^{z} \operatorname{artanh}(\tanh K_{0,i}X_i)\right] \\
X_i &= \tanh\left[B + \sum_{j=1}^{z-1} \operatorname{artanh}(\tanh K_{i,ij}X_{ij})\right] = \tanh L_i \\
X_{ij} &= \tanh\left[B + \sum_{k=1}^{z-1} \operatorname{artanh}(\tanh K_{ij,ijk}X_{ijk})\right] = \tanh L_{ij} \\
&\;\;\vdots \\
X_{i_1 i_2 \ldots i_N} &= \tanh B = \tanh L_{i_1 \ldots i_N}
\end{aligned} \tag{3.1}$$

and we wish to evaluate the mean value of $\langle \sigma_0 \rangle$ when the K's are independent random variables with Gaussian distribution[2]

$$P(K) = (z/2\pi\tilde{K}^2)^{1/2}\exp\left[-z(K - K_0 z^{-1})^2/2\tilde{K}^2\right] \tag{3.2}$$

Since we are particularly interested in the "classical" limit $z \to \infty$, we make the change of variables

$$K_\alpha = K_0 z^{-1} + \tilde{K} x_\alpha z^{-1/2} \tag{3.3}$$

in (3.1) and (3.2), where α denotes symbolically the set of ordered pairs of indices $(i_1 i_2 \ldots i_{s-1}; i_1 i_2 \ldots i_s)$, and consider in place of (3.1) the hierarchy

$$\begin{aligned}
\langle \sigma_0 \rangle &= \tanh\left(B + K_0 z^{-1} \sum_{i=1}^{z} X_i + \tilde{K} z^{-1/2} \sum_{i=1}^{z} x_i X_i \right) \\
X_i &= \tanh\left(B + K_0 z^{-1} \sum_{j=1}^{z} X_{ij} + \tilde{K} z^{-1/2} \sum_{j=1}^{z} x_{i,ij} X_{ij} \right) \\
X_{ij} &= \tanh\left(B + K_0 z^{-1} \sum_{k=1}^{z} X_{ijk} + \tilde{K} z^{-1/2} \sum_{k=1}^{z} x_{ij,ijk} X_{ijk} \right) \\
&\;\;\vdots \\
X_{i_1 i_2 \ldots i_N} &= \tanh B
\end{aligned} \tag{3.4}$$

which is easily obtained by substituting (3.3) into (3.1) and keeping only the leading order terms in the Taylor expansions of artanh y and tanh y for small y. Also, for convenience, we have allowed all indices to range from unity to z, which has no effect asymptotically in the limit $z \to \infty$.

The x_α are now independent random variables with Gaussian distribution

$$p(x) = (2\pi)^{-1/2}\exp(-x^2/2) \tag{3.5}$$

and the quantity we wish to calculate is the mean value

$$E\{\langle \sigma_0 \rangle\} \equiv \int \cdots_{-\infty}^{\infty} \int \langle \sigma_0 \rangle \prod_{(\alpha)} p(x_\alpha)\, dx_\alpha \tag{3.6}$$

Let us first consider that part of (3.6) defined by the integral

$$I = \int \cdots_{-\infty}^{\infty} \int \tanh\left(B + K_0 z^{-1} \sum_{i=1}^{z} X_i + \tilde{K} z^{-1/2} \sum_{i=1}^{z} x_i X_i \right) \prod_{i=1}^{z} p(x_i)\, dx_i \tag{3.7}$$

which essentially amounts to integrating over the first shell of the lattice.

Define a new variable u by

$$u = (Q_N z)^{-1/2} \sum_{i=1}^{z} x_i X_i \tag{3.8}$$

and choose Q_N such that the vector

$$\mathbf{X} = (Q_N z)^{-1/2}(X_1, X_2, \ldots, X_z) \tag{3.9}$$

has unit norm. That is

$$Q_N = z^{-1} \sum_{i=1}^{z} X_i^2 \tag{3.10}$$

We can then construct an orthogonal matrix A (for example by the Gram–Schmidt process) whose first row is $\mathbf{X}$. Using this matrix $A = (a_{ij})$ to make an orthogonal change of variables

$$y_i = \sum_{j=1}^{z} a_{ij} x_j, \qquad y_1 \equiv u \tag{3.11}$$

it then follows straightforwardly from (3.5), (3.7), and (3.8) that

$$I = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2} \tanh\big(B + K_0 M_N + \tilde{K} Q_N^{1/2} u\big)\, du \tag{3.12}$$

where

$$M_N = z^{-1} \sum_{i=1}^{z} X_i \tag{3.13}$$

We now need to evaluate the "average" of (3.12) over the second shell variables $x_{i,ij}$. Referring to (3.4) and repeating the above steps (3.8)–(3.12) we define new variables

$$u_i = (Q_{N-1} z)^{-1/2} \sum_{j=1}^{z} x_{i,ij} X_{ij}, \qquad i = 1, 2, \ldots, z \tag{3.14}$$

where

$$Q_{N-1} = z^{-1} \sum_{j=1}^{z} X_{ij}^2 \tag{3.15}$$

We can then write

$$X_i = \tanh\big(B + K_0 M_{N-1} + \tilde{K} Q_{N-1}^{1/2} u_i\big) \tag{3.16}$$

where

$$M_{N-1} = z^{-1} \sum_{j=1}^{z} X_{ij} \tag{3.17}$$

and after an appropriate orthogonal change of variables we can make the replacement

$$\prod_{j=1}^{z-1} p(x_{i,ij})\,dx_{i,ij} \mapsto p(u_i)\,du_i \tag{3.18}$$

in the average of I over second shell variables. [We have depressed the "i dependence" of Q_{N-1} and M_{N-1} in (3.15) and (3.17) since ultimately this dependence comes solely from dummy integration variables such as in (3.20) and (3.21).]

Repeating this process to the outermost shell we obtain a nested hierarchy of equations ending in

$$X_{i_1 \ldots i_{N-2}} = \tanh\left(B + K_0 M_2 + \tilde{K} Q_2^{1/2} u_{i_1 \ldots i_{N-2}}\right) \tag{3.19}$$

where

$$Q_2 = z^{-1} \sum_{i_{N-1}=1}^{z} \left[X(u_{i_1 \ldots i_{N-1}})\right]^2 \tag{3.20}$$

$$M_2 = z^{-1} \sum_{i_{N-1}=1}^{z} X(u_{i_1 \ldots i_{N-1}}) \tag{3.21}$$

and from the last of equations (3.4),

$$X(u_{i_1 \ldots i_{N-1}}) = \tanh\left(B + K_0 M_1 + \tilde{K} Q_1^{1/2} u_{i_1 \ldots i_{N-1}}\right) \tag{3.22}$$

with

$$M_1 = Q_1^{1/2} = \tanh B \tag{3.23}$$

The mean value (3.6) of $\langle \sigma_0 \rangle$ is then the Gaussian average of the integral I [Eq. (3.12)] with respect to the variables $u_{i_1 \ldots i_s}$, $s = 1, 2, \ldots, N-1$.

If we integrate first over the outermost shell we observe from (3.20) and (3.21) that we need to evaluate integrals of the form

$$J_z = \int \underset{-\infty}{\overset{\infty}{\cdots}} \int f\left(z^{-1} \sum_{i=1}^{z} g(y_i)\right) \prod_{i=1}^{z} p(y_i)\,dy_i \tag{3.24}$$

at least asymptotically in the limit $z \to \infty$. The evaluation of this limit is fairly straightforward and in fact is a consequence of the weak law of large numbers.[12] A brief discussion is given in Appendix A with the result that for bounded and continuous f and any normalized nonnegative distribution p such that $gp^{1/2}$ is square integrable,

$$\lim_{z \to \infty} J_z = f\left(\int_{-\infty}^{\infty} g(y) p(y)\,dy\right) \tag{3.25}$$

Using this result repeatedly from the outermost to the innermost shell (with $g = X$ and X^2) we obtain in the limit $z \to \infty$ an SK hierarchy of equations for the means of M_i and Q_i, given, respectively, by

$$m_i = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2} \tanh\left(B + K_0 m_{i-1} + \tilde{K} q_{i-1}^{1/2} u\right) du \qquad (3.26)$$

and

$$q_i = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2} \left[\tanh\left(B + K_0 m_{i-1} + \tilde{K} q_{i-1}^{1/2} u\right)\right]^2 du \qquad (3.27)$$

where

$$m_1 = q_1^{1/2} = \tanh B \qquad (3.28)$$

and from (3.6), (3.7), and (3.12) for an N shell Cayley tree with infinite coordination number,

$$E\{\langle \sigma_0 \rangle\} = m_{N+1} \quad \text{and} \quad E\{\langle \sigma_0 \rangle^2\} = q_{N+1} \qquad (3.29)$$

In the event that

$$m = \lim_{N \to \infty} m_N \quad \text{and} \quad q = \lim_{N \to \infty} q_N \qquad (3.30)$$

exist, it is clear that m and q must be fixed points of (3.26) and (3.27), in which case these equations reduce to the SK equations. The particular fixed points approached in this limit depend on the initial conditions (3.28) and we stress once more that in order to obtain nontrivial values for m and q, the field B must only be set equal to zero after the thermodynamic limit $N \to \infty$ has been taken.

For example, when $K_0 = 0$ and $B > 0$, (3.27) becomes a first-order difference equation for the q_i and it is not difficult to show in this case (see Appendix B) that the q_i approach the nontrivial fixed point $q > 0$ of (3.27). Moreover when $\tilde{K} > 1$, this fixed point remains positive in the limit $B \to 0$ and gives rise to the spin glass state $m = 0$ and $q > 0$.[(2)]

We stress again, however, that this state, which holds rigorously for the Cayley tree, must be interpreted as a local rather than a global state. Likewise, the mean free energy, given in terms of the quenched local magnetization $m = m(\beta, B)$ by[(8)]

$$\beta\psi(\beta, B) = \int_B^{\infty} \left[m(\beta, b) - 1\right] db - B + E_0 \qquad (3.31)$$

should be interpreted as a local free energy. Thus for example, when $K_0 = 0$, we show in Appendix C that (3.26), (32.7), and (3.31) yield the SK free energy

$$-\beta\psi(\beta, B) = \tilde{K}^2(1-q)^2/4 + \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-x^2/2} \log 2\cosh\left(B + \tilde{K} x q^{1/2}\right) dx \qquad (3.32)$$

where q is the positive fixed point of (3.27). This local free energy, as is well known,[2] gives negative entropy at low temperatures.

4. THE TAP EQUATIONS

Specializing to the case $K_0 = 0$ and recalling the expression (2.8) for the expectation value of a spin σ_j in the first shell, we find on substituting (3.3), that from (3.1) and (3.4),

$$\langle \sigma_j \rangle \sim X_j + \tilde{K} x_j z^{-1/2}(1 - X_j^2)\langle \sigma_0 \rangle, \qquad j \neq 0 \quad \text{as} \quad z \to \infty \tag{4.1}$$

where the X_j's are defined recursively by (3.4).

From (4.1) it is an easy matter to derive the TAP equations by noting first from (3.4) that

$$\langle \sigma_0 \rangle \sim \tanh\left(B + \tilde{K} z^{-1/2} \sum_{j=1}^{z} x_j X_j \right) \qquad \text{as } z \to \infty \tag{4.2}$$

and then by eliminating the X_j between (4.1) and (4.2), we obtain to leading order

$$\langle \sigma_0 \rangle = \tanh\left\{ B + \sum_{j=1}^{z} \tilde{K} x_j z^{-1/2} \langle \sigma_j \rangle - \langle \sigma_0 \rangle \sum_{j=1}^{z} \tilde{K} x_j^2 z^{-1} \left[1 - \langle \sigma_j \rangle^2 \right] \right\} \tag{4.3}$$

These equations are only a slightly disguised version of the TAP equations[5] with $\langle \sigma_j \rangle$ replacing m_j and $\tilde{K} x_j z^{-1/2}$ replacing βJ_{0j} in the usual notation.

Although the TAP equations have a certain physical appeal and interpretation it is not at all clear how one now proceeds with the averaging over the Gaussian random variables $x_{i_1 \ldots i_s}$ and thereafter to the classical limit $z \to \infty$. With the equivalent equations (4.1), however, the procedure is relatively straightforward as we now show.

Firstly, under the orthogonal change of variables (3.11), which can be inverted to give

$$x_i = \sum_{j=1}^{z} a_{ji} y_j \qquad (y_1 = u) \tag{4.4}$$

Eqs. (4.1) and (4.2) yield

$$\langle \sigma_j \rangle \sim X_j + \tilde{K} z^{-1/2} \sum_{i=1}^{z} a_{ij} y_i (1 - X_j^2) \tanh(B + \tilde{K} y_1 Q_N^{1/2}) \tag{4.5}$$

where Q_N is defined by (3.10). Since the first row of the matrix (a_{ij}) is the vector (3.9), we have

$$a_{1j} = X_j (Q_N z)^{-1/2} \tag{4.6}$$

Also, since the distribution of the Gaussian random variables x_i (and hence

y_i) is symmetric about the origin, $E\{y_i\} = 0$, $i = 1, 2, \dots, z$. It then follows from (4.5) that

$$E\left\{ z^{-1} \sum_{j=1}^{z} \langle \sigma_j \rangle \right\}$$
$$= E\left\{ z^{-1} \sum_{j=1}^{z} X_j \right\}$$
$$+ E\left\{ \tilde{K} z^{-2} Q_N^{-1/2} y_1 \sum_{j=1}^{z} X_j (1 - X_j^2) \tanh\left(B + \tilde{K} y_1 Q_N^{1/2}\right) \right\} \quad (4.7)$$

The second term on the right-hand side of (4.7) is $O(z^{-1})$ and by definition the left-hand side is the magnetization per spin of the first shell. It follows then from (3.13) and the argument leading to the SK hierarchy (3.26) and (3.27) that in the classical limit,

$$\lim_{z\to\infty} E\left\{ z^{-1} \sum_{j=1}^{z} \langle \sigma_j \rangle \right\} = \lim_{z\to\infty} E\{ M_N \} = m_N \quad (4.8)$$

where m_N is obtained recursively from (3.26) and (3.27).

In this way, the TAP equations, once the averaging and classical limit have been taken, reproduce the SK hierarchy (3.26) and (3.27) and provide the interpretation that m_i and q_i represent, respectively, the mean magnetization and the mean square magnetization of the ith shell from the surface of the lattice.[8]

5. DISCUSSION

In this paper we have studied the random Ising model on a Cayley tree rigorously in the limit of infinite coordination number $z \to \infty$.

An iterative scheme was developed which relates the mean magnetizations and their mean squares of successive shells far removed from the surface of the lattice. In this way we obtain *local* properties of the model by examining the asymptotic properties of the iterative scheme in the (thermodynamic) limit of an infinite number of shells.

For the particular case where the coupling constants are independent Gaussian random variables, the SK expressions for the mean magnetization and its mean square emerge as fixed points of our iterative scheme. These expressions and the corresponding SK free energy which we obtain by integration, should all be interpreted as local quantities so that the occurrence of negative *local* entropy at low temperatures is now mathematically possible although perhaps still physically undesirable. Nevertheless, these results are valid locally for the Cayley tree and provide a possible interpretation of the SK equations as a local mean-field theory for spin glasses.

Finally, we have reexamined the TAP equations which are based on the locally valid Bethe approximation on a Cayley tree. We have shown that if the average over bond disorder and the limit $z \to \infty$ are actually performed on these equations, one recovers our iterative scheme for shell magnetizations and their mean square and hence, in the thermodynamic limit, one also recovers the locally valid SK mean-field equations.

It should be mentioned that the actual convergence of our iterative scheme to the SK equations as stable fixed points has only been discussed here for the symmetric Gaussian distribution considered by TAP.

Alternative equations based on other types of bond distributions and general questions of convergence of associated iterative schemes will hopefully form the basis of a subsequent publication.

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APPENDIX A. DERIVATION OF EQUATION (3.25)

Let us first define the probability measure μ by

$$d\mu(x) = p(x)\,dx \tag{A.1}$$

and denote by $m(S)$, the measure of the set

$$S = \{(x_1, \ldots, x_z) : |G(\mathbf{x}) - G| > \epsilon\}$$

with respect to the z-dimensional measure

$$d\mu = d\mu(x_1) \ldots d\mu(x_z) \tag{A.2}$$

That is

$$m(S) = \int_S d\mu \tag{A.3}$$

where

$$G(\mathbf{x}) = z^{-1} \sum_{i=1}^{z} g(x_i) \quad \text{and} \quad G = \int_{-\infty}^{\infty} g(x)\,d\mu(x) \tag{A.4}$$

We first prove the following lemma which is an elementary generalization of the weak law of large numbers.[12]

Lemma. For square integrable g with respect to $d\mu$,

$$m(S) \leqslant (z\epsilon^2)^{-1} \int_{-\infty}^{\infty} [g(x) - G]^2\,d\mu(x) \tag{A.5}$$

Proof. Consider the integral

$$\int \overset{\infty}{\underset{-\infty}{\cdots}} \int \left[\sum_{i=1}^{z} g(x_i) - zG \right]^2 d\mu(x_1) \ldots d\mu(x_z) \geqslant \int_S \left[\sum_{i=1}^{z} g(x_i) - zG \right]^2 d\mu$$
$$\geqslant z^2 \epsilon^2 m(S) \qquad \text{(A.6)}$$

From (A.4) it is easily shown that the left-hand side of (A.6) is equal to

$$z \int_{-\infty}^{\infty} [g(x) - G]^2 d\mu(x)$$

from which the result (A.5) follows. ■

Assuming now that f is bounded and continuous we can, given $\delta > 0$, choose $\epsilon > 0$ such that

$$|G(\mathbf{x}) - G| < \epsilon \Rightarrow |f(G(\mathbf{x})) - f(G)| < \delta \qquad \text{(A.7)}$$

It then follows that

$$\left| \int \{ f(G(\mathbf{x})) - f(G) \} \, d\mu \right| \leqslant \int_{|G(\mathbf{x}) - G| < \epsilon} |f(G(\mathbf{x})) - f(G)| \, d\mu$$
$$+ \int_{|G(\mathbf{x}) - G| > \epsilon} |f(G(\mathbf{x})) - f(G)| \, d\mu$$
$$\leqslant \delta + 2Mm(S) \qquad \text{(A.8)}$$

where

$$M = \sup \{ f(x) \} < \infty$$

The required result (3.25) then follows from the lemma by taking $z \to \infty$ in (A.8) and noting that $\delta > 0$ is arbitrary.

APPENDIX B. STABILITY OF THE SK FIXED POINT

When $K_0 = 0$ the first-order difference equation (3.27) for $q_i^{1/2} \equiv Y_i$ can be expressed in the form

$$Y_{i+1} = F(Y_i), \qquad i = 1, 2, \ldots \qquad (Y_0 = 0) \qquad \text{(B.1)}$$

where

$$F(y) = \left\{ (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2} [\tanh(B + \tilde{K} y u)]^2 du \right\}^{1/2} \qquad \text{(B.2)}$$

Granted that there is at most one nonnegative solution of the SK equations,[2] and noting from (B.2) that $F(0) = \tanh B > 0$ and $F(y) \to 1$ as

$y \to \infty$, it follows that there is only one fixed point $0 < y^* < 1$ of (B.1) satisfying

$$y^* = F(y^*) \tag{B.3}$$

and such that

$$F'(y^*) < 1 \tag{B.4}$$

Moreover, from (B.2) we have in general that

$$\begin{aligned} F'(y) &= \left[2F(y)\right]^{-1}(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-u^2/2} 2\tanh(B + \tilde{K}yu) \\ &\quad \times \left[1 - \tanh^2(B + \tilde{K}yu)\right]\tilde{K}u\,du \\ &= \left[2yF(y)\right]^{-1}(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-u^2/2} u \frac{\partial}{\partial u}\left[\tanh^2(B + \tilde{K}yu)\right]du \\ &= \left[2yF(y)\right]^{-1}(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-u^2/2}(u^2 - 1)\tanh^2(B + \tilde{K}yu)\,du \end{aligned} \tag{B.5}$$

where in the last step we have integrated by parts.

It follows from (B.2) and (B.5) that at the fixed point (B.3),

$$\begin{aligned} F'(y^*) &= \frac{1}{2}\left\{\left[F(y^*)\right]^{-2}(2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{-u^2/2}u^2\tanh^2(B + \tilde{K}y^*u)\,du - 1\right\} \\ &\geqslant -1/2 \end{aligned} \tag{B.6}$$

Equations (B.4) and (B.6) together imply that y^* is stable, so that in (B.1) $Y_i \to y^*$ as $i \to \infty$.

APPENDIX C. DERIVATION OF THE LOCAL FREE ENERGY (3.31)

In general, the local free energy ψ^* for an Ising model with zero field interaction energy $E\{\sigma\}$, is given in terms of the local magnetization m^* by[8]

$$\beta\psi^*(\beta, B) = \int_B^{\infty}\left[m^*(\beta, b) - 1\right]db + E_0 - B \tag{C.1}$$

where E_0 is the limiting value of $E\{\sigma\}$ per spin when all spins are set equal to unity.

In the case of a symmetric bond probability distribution function such as (3.2), with K_0 equal to zero, the average of E_0 is zero and the quenched local free energy ψ and magnetization m are related by

$$\beta\psi(\beta, B) = \int_B^{\infty}\left[m(\beta, b) - 1\right]db - B \tag{C.2}$$

Assume now that m and its mean square q are (nonnegative) fixed points of (3.26) and (3.27) (with $K_0 = 0$). That is

$$m(\beta, b) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2} \tanh(b + \tilde{K}q^{1/2}u)\, du \tag{C.3}$$

and

$$q(\beta, b) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2} \left[\tanh(b + \tilde{K}q^{1/2}u)\right]^2 du \tag{C.4}$$

Noting that $2\cosh\alpha \sim e^{\alpha}$ and $q(\beta, b) \sim 1$ as α and b tend to infinity, we arrive at the identity

$$\begin{aligned}
B &- \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-u^2/2} \log 2\cosh(B + \tilde{K}q^{1/2}u)\, du \\
&= \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-u^2/2} \left\{ \int_{B}^{\infty} \left[\frac{\partial}{\partial b} \log 2\cosh(b + \tilde{K}q^{1/2}u) - 1 \right] db \right\} du \\
&= \int_{B}^{\infty} \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-u^2/2} \left\{ \tanh(b + \tilde{K}q^{1/2}u) - 1 \right\} du\, db \\
&\quad + \int_{B}^{\infty} \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-u^2/2} \tilde{K}u \tanh(b + \tilde{K}q^{1/2}u) \frac{\partial q^{1/2}}{\partial b}\, du\, db
\end{aligned} \tag{C.5}$$

The first term on the right-hand side of (C.5) is easily seen from (C.2) and (C.3) to be

$$\int_{B}^{\infty} \left[m(\beta, b) - 1 \right] db = B + \beta\psi(\beta, B) \tag{C.6}$$

To simplify the second term, note from (C.4) that

$$\begin{aligned}
1 - q &= \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-u^2/2} \left[1 - \tanh^2(b + \tilde{K}q^{1/2}u) \right] du \\
&= (2\pi q\tilde{K}^2)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2} \frac{\partial}{\partial u} \left[\tanh(b + \tilde{K}q^{1/2}u) \right] du \\
&= (2\pi q\tilde{K}^2)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2} u \tanh(b + \tilde{K}q^{1/2}u)\, du
\end{aligned} \tag{C.7}$$

The second term on the right-hand side of (C.5) can then be written as

$$\begin{aligned}
&\int_{B}^{\infty} (\tilde{K}^2/2) \frac{\partial q}{\partial b} \left\{ (2\pi q\tilde{K}^2)^{-1/2} \int_{-\infty}^{\infty} e^{-u^2/2} u \tanh(b + \tilde{K}q^{1/2}u)\, du \right\} db \\
&= \int_{B}^{\infty} (\tilde{K}^2/2)(1 - q) \frac{\partial q}{\partial b}\, db \\
&= (\tilde{K}^2/2) \int_{B}^{\infty} \frac{\partial}{\partial b} (q - q^2/2)\, db \\
&= \tilde{K}^2 (1 - q)^2/4
\end{aligned} \tag{C.8}$$

Combining (C.5), (C.6) and (C.8), we have the SK expression for the local free energy

$$-\beta\psi(\beta, B) = \frac{\tilde{K}^2(1-q)^2}{4} + \int_{-\infty}^{\infty} (2\pi)^{-1/2} e^{-u^2/2} \log 2\cosh(B + \tilde{K}q^{1/2}u)\, du \tag{C.9}$$

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